9
Dimensional analysis, similitude and hydraulic models

9.1 Introduction

Exact theoretical solutions to fluid flow problems are generally only available for laminar flow conditions and simple boundary conditions, circumstances rarely found in civil engineering. Recourse to experiment may be necessary, especially where the physical boundaries are complex. One of the difficulties which face the analyst is the large number of variables which may influence a particular flow phenomenon. However, by judicious grouping of the variables involved into composite variable groups, it is possible to reduce the number of variables used to define a particular flow problem. This can be accomplished by application of the principle that, in a physically correct equation, all terms must have the same dimensions. The primary dimensions which characterise fluid flow systems are mass M, length L and time T. All system parameters, such as force, power, velocity etc. can be expressed in MLT terms:

force = mass x acceleration = MLT^{-2}
power = force x velocity = $ML^{2}T^{-3}$
dynamic viscosity = force x time/area = $ML^{-1}T^{-1}$

9.2 Dimensionless quantities

Each fluid flow characteristic is dependent on a number of variables. For example, the force F in a particular flow environment can be expressed in the form

$$F = \phi(v, d, \rho, \mu) \qquad (9.1)$$

where ϕ means 'function of'. Any such functions can be represented as a power series sum:

$$F = v^{a1} + d^{b1}\rho^{c1}\mu^{d1} + v^{a2}d^{b2}\rho^{c2}\mu^{d2} + \ldots\ldots$$

where a1, b1, a2, b2, … are numerical indices. Dividing across by the first term on the right-hand side:

$$\frac{F}{v^{a1}d^{b1}\rho^{c1}\mu^{d1}} = 1 - v^{a2-a1}d^{b2-b1}\rho^{c2-c1}\mu^{d2-d1} + \ldots\ldots$$

Since the first term on the right hand side is dimensionless then all terms in the equation must be non-dimensional, that is,

$$\left[\frac{F}{v^{a}d^{b}\rho^{c}\mu^{d}}\right] = 0 \qquad (9.2)$$

where [ ] indicates "dimensions of".

9.3 The Buckingham π theorem

A phenomenon which is a function of n variables can be modelled as follows:

$$\phi(x_1, x_2, x_3, \ldots, x_n) = 0$$

Such a phenomenon can also be described as a function of (n-m) non-dimensional group variables, where m is the number of basic component dimensions of the variables $x_1, \ldots, x_n$. In fluid flow these basic dimensions are M, L and T, so that m = 3. The corresponding non-dimensional functional relationship is

$$F(\pi_1, \pi_2, \pi_3, \ldots\ldots\ldots, \pi_{n-m}) = 0$$

Each π term is a non-dimensional grouping of (m+1) variables, m of which are repeated in all terms. For example, the pressure drop in pipe flow can be expressed as a function of six variables:

$$\phi(\Delta p, L, \rho, v, D, \mu, k) = 0 \qquad (9.3)$$

where Δp is the pressure drop over a pipe length L and k is the pipe wall roughness. In this case n = 7 and m = 3. Taking v, ρ and D as the three repeated variables, the alternative non-dimensional functional relationship is:

$$F(\pi_1, \pi_2, \pi_3, \pi_4) = 0$$

Each π term is a grouping of 4 (that is m+1) variables, 3 (that is m) of which are repeated in all π terms. Taking the first term π_1:

$$\pi_1 = v^{\alpha} \rho^{\beta} D^{\gamma} \Delta p$$

and is non-dimensional. Hence

$$\left[\left(LT^{-1}\right)^{\alpha}\left(ML^{-3}\right)^{\beta} L^{\gamma} ML^{-1}T^{-2}\right] = 0$$

from which it follows that $\alpha = -2$, $\beta = -1$ and $\gamma = 0$, giving π_1 the following value:

$$\pi_1 = \frac{\Delta p}{\rho v^2}$$

Similarly

$$\pi_2 = \frac{L}{D}, \qquad \pi_3 = \frac{\mu}{\rho v D} \qquad \text{and} \qquad \pi_4 = \frac{k}{D}$$

The resulting non-dimensional functional relationship for pressure drop in pipe flow is

$$F\left(\frac{\Delta p}{\rho v^2}, \frac{L}{D}, \frac{\mu}{\rho v D}, \frac{k}{D}\right) = 0 \qquad (9.4)$$

The number of variables has been reduced from seven to four. The non-dimensional group variables can be combined by multiplication or division:

$$F\left(\frac{\Delta p\, D}{L \rho v^2}, \frac{\mu}{\rho v D}, \frac{k}{D}\right) = 0$$

Hence

$$\frac{\Delta p}{L} = \frac{\rho v^2}{D} \phi\left(\frac{k}{D}, R_e\right) \qquad (9.5)$$

In the Darcy-Weisbach equation for pipe flow the friction factor f is related to pressure drop as follows:

$$\frac{\Delta p}{L} = \frac{\rho v^2}{D} f$$

Hence f = φ(k/D, R_e), as in the Colebrook-White equation. In this case, however, the derivation has been based on dimensional reasoning and a judicious selection of the three repeated variables v, D and ρ.

9.4 Physical significance of non-dimensional groups

The force components in fluid systems arise from gravity, viscosity, elasticity, surface tension and pressure influences. The resultant force is called the inertial force (F_i) and the ratio of each of the above force components to the resultant force indicates the relative significance of each on overall system behaviour.

1. Gravity

$$\frac{F_i}{F_g} = \frac{MLT^{-2}}{Mg} = \frac{v^2}{gL} = F_r^{\,2}$$

where F_r is the Froude number.

2. Viscosity

$$\frac{F_i}{F_m} = \frac{MLT^{-2}}{\mu L^2 T^{-1}} = \frac{\rho L^4 T^{-2}}{\rho L^2 T^{-1}} = \frac{\rho L v}{\mu} = R_e$$

where R_e is the Reynolds number.

3. Surface tension

$$\frac{F_i}{F_\sigma} = \frac{MLT^{-2}}{\sigma L} = \frac{\rho L^4 T^{-2}}{\sigma L} = \frac{\rho L v^2}{\sigma} = W_e^{\,2}$$

where W_e is the Weber number.

9.5 Similarity requirements in model studies

Dynamic similarity between model and prototype requires that the ratios of the inertial force to its individual force components are the same in model and prototype. This implies that the Reynolds, Froude and Weber numbers have the same values in model and prototype.

Geometric similarity is assured by adopting a fixed scale ratio for all dimensions.

If model and prototype are dynamically and kinematically similar then the flow patterns will be the same at both scales, resulting in kinematic similarity.

1. R_e similarity

$$\left(\frac{vL\rho}{\mu}\right)_m = \left(\frac{vL\rho}{\mu}\right)_p$$

where the subscripts m and p relate to model and prototype, respectively.

2. F_r similarity

$$\left(\frac{v}{\sqrt{gL}}\right)_m = \left(\frac{v}{\sqrt{gL}}\right)_p$$

If $g_m = g_p$ then

$$\frac{v_m}{v_p} = \left(\frac{L_m}{L_p}\right)^{0.5}$$

Thus if the same fluid is used in model and prototype (that is ρ and μ are the same), it is not possible to achieve complete similarity because of the conflicting operational requirements for Re and Fr similarity. In practice, a compromise is reached by basing scaling relationships on the predominant force component. For flows without a free surface, for example, pipe flow and flow around submerged bodies such as submarine, aircraft, motor vehicles and buildings, R_e is taken as the scaling criterion. For free surface phenomena, for example, hydraulic structures, ships and so on, Fr scaling is used. An illustrative example of F_r scaling is shown on Fig 9.1.

insert Fig 9.1

Fig 9.1 Plate showing dam spillway model, Scale 1:50 Hydraulics Laboratory, University College Dublin, Courtesy A L Dowley, by permission of K. O'Donnell Chief Engineer, Dublin Corporation.

Where ρ, μ and g are assumed to be the same in model and prototype, the scale ratios for the various flow parameters, as determined by R_e and F_r scaling, can be expressed in terms of the length scale ratio λ, where $\lambda = L_m/L_p$. Table 9.1 shows scale ratios for flow variables.

The influence of surface tension (Weber number) is generally not significant in hydraulic model studies - refer to related comments in Section 9.5.2.

Table 9.1
Scale ratios for flow variables

Variable	Dimensions	R_e scaling	F_r scaling
Time	T	λ^2	$\lambda^{0.5}$
Velocity	LT^{-1}	λ^{-1}	$\lambda^{0.5}$
Acceleration	LT^{-2}	λ^{-3}	λ^0
Discharge	L^3T^{-1}	λ	$\lambda^{2.5}$
Force	ρL^4T^{-2}	λ^0	λ^3
Pressure	ρL^2T^{-2}	λ^{-2}	λ
Power	ρL^5T^{-3}	λ^{-1}	$\lambda^{3.5}$

9.5.1 Pumps and turbines

Discharge

The discharge rate Q through a pump or turbine can be expressed in terms of the geometric characteristics of the device, the operating head and the fluid properties, in the general functional form

$$f(Q, N, D, B, gH, \rho, \mu) = 0 \qquad (9.6)$$

where N is the rotational speed, D and B are the impeller or runner diameter and width, respectively, and GHQ is a measure of the operating pressure rise/drop across the device. Replacing this functional

relationship with its non-dimensional equivalent by taking r, N and D as the repeated variables in the transformation procedure, yields the following:

$$\phi\left(\frac{B}{D}, \frac{Q}{ND^3}, \frac{gH}{N^2D^2}, \frac{\mu}{\rho ND^2}\right) \tag{9.7}$$

If the numerical values of the non-dimensional groups in model and prototype are equal, then complete similarity is achieved. The ratio B/D infers geometric similarity; the non-dimensional groups gH/N^2D^2 and $\mu/\rho ND$ can be recognised as Fr^{-2} and Re^{-1}, respectively.

Power

Similarly, pump or turbine power P can be expressed in a form similar to eqn (9.6):

$$f(P, N, D, B, gH, \rho, \mu) = 0 \tag{9.8}$$

Using ρ, N and D as the repeated variables, the corresponding non-dimensional relation is found:

$$\phi\left(\frac{P}{\rho N^3D^5}, \frac{B}{D}, \frac{gH}{N^2D^2}, \frac{\mu}{\rho ND^2}\right) = 0 \tag{9.9}$$

Specific speed

It follows from equation (9.7) that, if the viscosity influence is neglected, dynamic similaraity is achieved in geometrically similar pumps, if the following relationships are satisfied:

$$\left(\frac{Q}{ND^3}\right)_m = \left(\frac{Q}{ND^3}\right)_p, \qquad \text{hence} \qquad \frac{N_m}{N_p}\left(\frac{D_p}{D}\right)^3$$

and

$$\left(\frac{N^2D^2}{gH}\right)_m = \left(\frac{N^2D^2}{gH}\right)_p, \qquad \text{hence} \qquad \left(\frac{D_p}{D_m}\right)^3 = \left(\frac{(gH)_p^{\,1.5}}{(gH)_m^{\,1.5}}\right)\left(\frac{N_m}{N_p}\right)^3$$

From these relations it follows that

$$\frac{N_m}{N_p} = \left(\frac{Q_p}{Q_m}\right)^{0.5}\left(\frac{(gh)_m}{(gH)_p}\right)^{0.75} \tag{9.10}$$

If the model is defined as having unit values of Q and H, the rotational speed of the model can be expressed, using equation (9.10), as

$$N_m = \frac{N_p(Q_p)^{0.5}}{(gH)_p^{\,0.75}} \tag{9.11}$$

The model speed, thus defined, is known as the ‘specific speed’ N_s:

$$N_s = \frac{NQ^{0.5}}{(gH)^{0.75}} \tag{9.12}$$

In the form presented in equation (9.12), the specific speed is a non-dimensional index, which can be used to categorise pump types, as discussed in chapter 11. In pump technology literature, the gravity constant is often omitted from the specific speed expression, resulting in the following dimensional form of the specific speed characteristic:

$$N_s = \frac{NQ^{0.5}}{H^{0.75}} \qquad (9.13)$$

9.5.2 The use of distorted scales

For practical reasons it may be desirable to use different vertical and horizontal scales. In rivers and estuaries the horizontal dimensions of the reach to be modelled may be very large relative to the water depth. In order to accommodate a model within a reasonable plan area it is often necessary to select an horizontal scale that is smaller than the vertical scale. The vertical scale should, as a general rule be not less than 1:100 and should not lead to water depths that are likely to be significantly influenced by surface tension effects.

Distorted scales influence scale relationships. Since the circumstances in which they are necessary invariably involve free surface flow, scale relationships are governed by Froude law scaling. Representing the horizontal scale as λ_x and the vertical as λ_y, the scale relations for velocity v and discharge Q, as dictated by the Froude number are as follows:

velocity = f(y), hence $\dfrac{v_m}{v_p} = \lambda_y^{0.5}$

discharge = f(v,x,y), hence $\dfrac{Q_m}{Q_p} = \lambda_y^{0.5}\lambda_x\lambda_y$

9.6 Concluding comments

Dimensional analysis is a valuable aid to modelling of flow phenomena. It enables the effects of a number of variables to be considered together. However, it does not yield any information on whether particular variables are important or not - this knowledge must be obtained from a physical examination of the problem.

Related reading

Allen, J (1952) Scale Models in Hydraulic Engineering, Longmans, London.
Francis, J. R. D. and Minton, P. (1984) Civil Engineering Hydraulics, Edward Arnold, London.
Novak, P. and Cabelka, J. (1981) Models in Hydraulic Engineering, Spon, London.
Featherstone, R. E. and Nalluri, C. (1982) Civil Engineering Hydraulics, Collins, London.